J. Math. Anal. Appl. 362 (2010) 224–230

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Application of the Pfaffian technique to the KR and mNVN equations

Metin Ünal

University of Uşak, Faculty of Science, Department of Mathematics, Uşak, Turkey

ARTICLE INFO

Article history:
Received 16 September 2008
Available online 7 August 2009
Submitted by G. Bluman

Keywords:
Pfaffian
Integrable systems

ABSTRACT

In this paper we discuss the differentiation of Pfaffians which is useful in Pfaffian technique. We apply the Pfaffian technique to two integrable systems; the Konopelchenko–Rogers (KR) equations and the modified Novikov–Veselov–Nizik (mNVN) equations and show that these equations in the bilinear form reduce to a Pfaffian identity. We also derive a new Lax pair for the mNVN equations which is gauge equivalent to a pair of operators.

1. Introduction

We study two $(2+1)$-dimensional integrable nonlinear evolution equations; both have dimensional reductions to known integrable equations in $(1+1)$-dimensions. If the two spatial variables appear on an equal footing and hence allow such reductions in either variable one calls the $(2+1)$-dimensional system a strong generalization of the $(1+1)$-dimensional system. For example, the KdV equation has two generalizations to $(2+1)$-dimensions, namely the KP equation which is a weak generalization, and the NVN equations which are a strong generalization of the KdV equation. The Konopelchenko–Rogers (KR) equations are a $(2+1)$-dimensional strong generalization of the $(1+1)$-dimensional sine-Gordon (sG) equation analogous to the modified Novikov–Veselov–Nizik (mNVN) equations

$$\begin{aligned}
&4u_t = u_{xxx} - u_{yyy} + 3u_x v_{xx} - 3u_y v_{yy} - u_x^3 + u_y^3,\\
&v_{xy} = u_x u_y.
\end{aligned}$$

These reduce to the potential mKdV equation

$$2u_t = u_{xxx} + 2u_x^3$$

when $y \to -x$. Hence the mNVN equations are a strong generalization of the potential mKdV equation. Both the KR and mNVN equations have Pfaffian solutions.

This paper is organized as follows. In Section 2, we introduce some properties of Pfaffians. In Section 3 we recall some results from [5] and [4]; we apply the gauge transformation to the Lax pair of the KR equations and, after rescaling the Lax pair, we derive the weak Lax pair for the KR equations. The compatibility of this Lax pair leads to the KR equations. In Section 4 we carry out the same procedure as for the two-dimensional sine-Gordon equation and obtain a new result for the mNVN equations. In Section 5 we also show that the KR and mNVN equations in the bilinear form reduce to the identity of Pfaffians.

E-mail address: drmetintr@yahoo.co.uk.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.07.060

2. Properties of Pfaffians

Roughly speaking, a Pfaffian is the square root of the determinant of a skew-symmetric matrix. Let

$$A = (a_{ij})$$

be an $n \times n$ skew-symmetric matrix (i.e. $a_{ij} = -a_{ji}$ and consequently $a_{ii} = 0$ for $i, j = 1, 2, \dots, n$). It is known that if n is odd, then $\det(A)$ is zero, but if n is even $\det(A)$ is a perfect square of a polynomial in the entries a_{ij}, called the *Pfaffian* of A and denoted by $\mathrm{Pf}(A)$. To be precise, for even n

$$\mathrm{Pf}(A) = \sum_{\sigma} \epsilon(\sigma) a_{\sigma(1),\sigma(2)} \cdots a_{\sigma(n-1),\sigma(n)},$$

where σ runs over the permutations of $\{1, \dots, n\}$ such that

$$\sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \dots, \sigma(n-1) < \sigma(n),$$
$$\sigma(1) < \sigma(3) < \cdots < \sigma(n-1),$$

and $\epsilon(\sigma)$ $(= \pm 1)$ is the parity of this permutation.

A classical notation for the Pfaffian of A [1] is

$$\mathrm{Pf}(A) = (1, 2, \dots, n),$$

where $(i, j) = a_{ij}$. One expansion rule for Pfaffians is given by

$$(1, 2, \dots, n) = \sum_{i=2}^{n} (-1)^i (1, i)(2, 3, \dots, \hat{i}, \dots, n),$$

where $\hat{}$ indicates that the index underneath should be deleted. For example for $n = 4$ we can write the Pfaffian representation as

$$(1, 2, 3, 4) = (1, 2)(3, 4) - (1, 3)(2, 4) + (1, 4)(2, 3).$$

2.1. Identities of Pfaffians

Identities of Pfaffians correspond to the Jacobi identity of determinants. The Jacobi identity is given as follows: for an $N \times N$ matrix A, we write $A^{i,\dots,j}_{k,\dots,l}$ for the minor obtained by omitting the ith, $\dots$, jth rows and the kth, $\dots$, lth columns, in this notation the Jacobi identity is

$$|A| A^{i,j}_{k,l} = \begin{vmatrix} A^i_k & A^j_k \\ A^i_l & A^j_l \end{vmatrix}.$$

Let m and n be positive integers. For the even case (even number of a_i) we have the following Pfaffian identity

$$\begin{aligned} &(a_1, a_2, \dots, a_{2m}, 1, 2, \dots, 2n)(1, 2, \dots, 2n) \\ &\quad = \sum_{s=2}^{2m} (-1)^s (a_1, a_s, 1, 2, \dots, 2n)(a_2, a_3, \dots, \hat{a}_s, \dots, a_{2m}, 1, 2, \dots, 2n), \end{aligned} \tag{1}$$

and for the odd case (odd number of a_i)

$$\begin{aligned} &(a_1, a_2, \dots, a_{2m-1}, 1, 2, \dots, 2n-1)(1, 2, \dots, 2n) \\ &\quad = \sum_{s=1}^{2m-1} (-1)^{s-1} (a_s, 1, 2, \dots, 2n-1)(a_1, a_2, \dots, \hat{a}_s, \dots, a_{2m-1}, 1, 2, \dots, 2n), \end{aligned} \tag{2}$$

where a_i are just extra indices in the same way that the a, b, c^T, d^T are extra columns and rows. (See [2] for the proof of the identities (1), (2).)

For example from (1) and (2), for $m = 2$, we have the following Pfaffian identities

$$\begin{aligned} (a_1, a_2, a_3, a_4, 1, 2, \dots, 2n)(1, 2, \dots, 2n) = {} & (a_1, a_2, 1, 2, \dots, 2n)(a_3, a_4, 1, 2, \dots, 2n) \\ & - (a_1, a_3, 1, 2, \dots, 2n)(a_2, a_4, 1, 2, \dots, 2n) \\ & + (a_1, a_4, 1, 2, \dots, 2n)(a_2, a_3, 1, 2, \dots, 2n) \end{aligned}$$

and

$$\begin{aligned}(a_1, a_2, a_3, 1, 2, \dots, 2n-1)(1, 2, \dots, 2n) &= (a_1, 1, 2, \dots, 2n-1)(a_2, a_3, 1, 2, \dots, 2n)\\ &\quad - (a_2, 1, 2, \dots, 2n-1)(a_1, a_3, 1, 2, \dots, 2n)\\ &\quad + (a_3, 1, 2, \dots, 2n-1)(a_1, a_2, 1, 2, \dots, 2n).\end{aligned}$$

2.2. Differentiation of Pfaffians

In this section we will show how the derivatives of Pfaffians may be represented by the sum of Pfaffians. Suppose that

$$(i, j)_x = g(\theta_i) f(\theta_j) - f(\theta_i) g(\theta_j),$$

where f, g are differential operators, then by defining indices f and g such that $(f, i) = f(\theta_i)$, $(g, i) = g(\theta_i)$ and $(f, g) = 0$, we have

$$(i, j)_x = (f, g, i, j) = \begin{vmatrix} (f, g) & (f, i) & (f, j) \\ & (g, i) & (g, j) \\ & & (i, j) \end{vmatrix} = \begin{vmatrix} 0 & f(\theta_i) & f(\theta_j) \\ & g(\theta_i) & g(\theta_j) \\ & & (i, j) \end{vmatrix} = g(\theta_i) f(\theta_j) - f(\theta_i) g(\theta_j).$$

In general, it can be shown that [2]

$$\frac{\partial}{\partial x}(1, 2, \dots, 2n) = (f, g, 1, 2, \dots, 2n).$$

Higher order derivatives of Pfaffians can be calculated in a similar way.

3. The two-dimensional sine-Gordon equation

The system of Konopelchenko and Rogers [3] arises as the compatibility conditions for the triad of operators

$$\begin{aligned}L_1 &= \partial_X + S\partial_Y,\\ L_2 &= \partial_t\partial_Y - V\partial_Y - W_Y,\\ L_3 &= \partial_t\partial_X - V\partial_X - W_X,\end{aligned}$$

where $\partial_z = \frac{\partial}{\partial z}$ is a derivative with respect to the indicated variable and

$$\begin{aligned}S &= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},\\ V &= \frac{1}{2}\begin{pmatrix} 0 & -\theta_t \\ \theta_t & 0 \end{pmatrix},\\ W_Y &= -\frac{1}{2\sin\theta}\begin{pmatrix} \phi_X - \cos\theta\phi_Y & \tilde{\phi_Y}\sin\theta \\ -\phi_Y\sin\theta & -(\tilde{\phi_X} + \cos\theta\tilde{\phi_Y}) \end{pmatrix},\\ W_X &= -\frac{1}{2\sin\theta}\begin{pmatrix} \phi_Y - \cos\theta\phi_X & \tilde{\phi_X}\sin\theta \\ -\phi_X\sin\theta & -(\tilde{\phi_Y} + \cos\theta\tilde{\phi_X}) \end{pmatrix},\end{aligned} \tag{3}$$

in which $\theta_t = \phi + \tilde{\phi}$.

Next we will transform the Lax pair of the 2-dimensional sine-Gordon (sG) equation into a new Lax pair which is gauge equivalent to a pair of operators. The Lax pair of the 2-dimensional sine-Gordon or Konopelchenko–Rogers (KR) equations is

$$\begin{aligned}L &:= g^{-1}L_1 g = \partial_X - J\partial_Y + Q,\\ M &:= g^{-1}(L_2 - JL_3)g = \partial_t\partial_Y + S\partial_t\partial_X + \frac{1}{2}(\theta_Y A + \partial_X SA)\partial_t + \frac{1}{2}(\theta_t\theta_Y + \theta_{Yt}A + \theta_{Xt}SA) + \tilde{W}_1 + \tilde{W}_2,\end{aligned}$$

where g is the gauge and

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

is the reflection matrix,

$$Q = \frac{1}{2}\begin{pmatrix} 0 & \theta_X + \theta_Y \\ \theta_Y - \theta_X & 0 \end{pmatrix},$$
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\tilde{W_1} = \frac{\theta_{Xt}}{4\cos\frac{\theta}{2}}\begin{pmatrix} \sin\frac{3\theta}{2} & -\cos\frac{3\theta}{2} \\ -\cos\frac{3\theta}{2} & -\sin\frac{3\theta}{2} \end{pmatrix},$$
$$\tilde{W_2} = \frac{1}{4}\begin{pmatrix} (\rho_X + \theta_{Yt} - \rho_Y)\tan\frac{\theta}{2} & (\rho_X - \theta_{Yt} + \rho_Y) \\ (-\rho_X + \theta_{Yt} + \rho_Y) & (\rho_X + \theta_{Yt} + \rho_Y)\tan\frac{\theta}{2} \end{pmatrix}, \tag{5}$$

in which $\rho = \phi - \tilde{\phi}$. The matrix $S = S_\theta$ can be written in the following form

$$S_\theta = -R_\theta J = -JR_{-\theta},$$

where

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{6}$$

is the rotation matrix and J is the reflection matrix given in (4). The gauge is chosen to be a 'half-rotation and reflection', thus $g := S_{\theta/2}$, for which $g^2 = I$. Next we rotate axes $(X, Y) \to (x, y)$ so that $\partial_x = \partial_X + \partial_Y$, $\partial_y = \partial_X - \partial_Y$ and in order to simplify the notation, and free θ for more conventional usage, we rescale $\theta \to 2u$ and $\Theta \to v_t$, where

$$\Theta_x = -\frac{\rho_y + \theta_{yt}\cos\theta}{2\sin\theta}, \qquad \Theta_y = \frac{\rho_x - \theta_{xt}\cos\theta}{2\sin\theta}.$$

In these variables the Lax pair is

$$L = \begin{pmatrix} \partial_x & u_x \\ -u_y & \partial_y \end{pmatrix}, \tag{7}$$
$$M = \begin{pmatrix} \partial_y\partial_t + v_{yt} & u_y\partial_t \\ -u_x\partial_t & \partial_x\partial_t + v_{xt} \end{pmatrix}. \tag{8}$$

Let

$$\Phi = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \tag{9}$$

be a common solution of $L\Phi = M\Phi = 0$ for L and M given by (7) and (8) respectively. If we write $L\Phi = 0$ and $M\Phi = 0$ in component form we get the linear equations for the KR equations

$$\phi^1_x + u_x\phi^2 = 0, \tag{10}$$
$$\phi^2_y - u_y\phi^1 = 0, \tag{11}$$
$$\phi^1_{yt} + v_{yt}\phi^1 + u_y\phi^2_t = 0, \tag{12}$$
$$\phi^2_{xt} + v_{xt}\phi^2 - u_x\phi^1_t = 0. \tag{13}$$

After eliminating the operators and using the linear equations (10)–(13) in the commutator for (7) and (8), we get

$$[L, M] = LM - ML = \begin{pmatrix} [L, M]_{1,1} & [L, M]_{1,2} \\ [L, M]_{2,1} & [L, M]_{2,2} \end{pmatrix},$$

where

$$[L, M]_{1,1} = v_{xyt} - u_y u_{xt} - u_x u_{yt},$$
$$[L, M]_{1,2} = -u_x v_{yt} - u_{xyt} - u_y v_{xt},$$
$$[L, M]_{2,1} = u_y v_{xt} + u_{xyt} + u_x v_{yt},$$
$$[L, M]_{2,2} = v_{xyt} - u_x u_{yt} - u_y u_{xt}.$$

Solving these equations for the compatibility condition $[L, M] = 0$, we get the KR equations

$$u_{xyt} + u_x v_{yt} + u_y v_{xt} = 0, \tag{14}$$
$$v_{xy} - u_x u_y = 0. \tag{15}$$

4. Lax pair for the modified Novikov–Veselov–Nizik equations

It is known that a Lax pair that gives Pfaffian solutions of nonlinear equations satisfies the following equation [6]

$$\partial_X L + L^\dagger \partial_X = 0, \tag{16}$$

where L is one of the Lax pair and $L^\dagger$ is the adjoint of L. As shown below this is satisfied by L_1 for the KR equations. We wish to study similar Lax pairs which obey this constraint.

We take the following Lax pair

$$\tilde{L} = \partial_Y + S\partial_X,$$
$$\tilde{M} = \partial_t + \left(E\partial_X^2 + \partial_X^2 F + T\partial_X + \partial_X H + B\right)\partial_X,$$

where S is given in (3) and E, F, T, H, B are arbitrary 2×2 matrices, which we will find out later. The adjoint pair is

$$\tilde{L}^\dagger = -\partial_Y - \partial_X S^T,$$
$$\tilde{M}^\dagger = -\partial_t - \partial_X\left(\partial_X^2 E^T + F^T\partial_X^2 - \partial_X T^T - H^T\partial_X + B^T\right).$$

Next we substitute these into the left-hand side of (16). We get $F = E^T$, $H = -T^T$ and $B = B^T$. The compatibility condition for this Lax pair $[\tilde{L}, \tilde{M}] = 0$ gives $E = S$ and $T = -T^T$. Hence we can write the Lax pair (strong) [6] for the mNVN equations in the following form:

$$\tilde{L} = \partial_Y + S\partial_X, \tag{17}$$
$$\tilde{M} = \partial_t + \left(S\partial_X^2 + \partial_X^2 S + T\partial_X + \partial_X T + B\right)\partial_X, \tag{18}$$

where S is given in (3), $T = wA$ is a skew-symmetric matrix in which A is given by (5), and B is a symmetric real matrix. To find the entries of the matrices T and B, we make use of the compatibility condition of the Lax pair. Before looking at this in more detail we determine the properties and the necessary derivatives of the matrices S and A.

In order that the compatibility condition $[\tilde{L}, \tilde{M}] = 0$ is satisfied, the coefficients of the operators ∂_X, ∂_X^2, ∂_X^3 must vanish. The coefficient of ∂_X^3 vanishes if

$$S_Y - SS_X - S_X S + ST - TS = SA(\theta_Y + 2w) = 0.$$

Hence $w = -\frac{1}{2}\theta_Y$ and $T = -\frac{1}{2}\theta_Y A$. From the coefficient of ∂_X^2, we get

$$(2\theta_{XX} + \theta_{YY})A = SB - BS \tag{19}$$

and, from the coefficient of ∂_X, we get

$$(\theta_{XXY} - 2\theta_t)SA - 3\theta_X\theta_{XY}S + 2B_Y + 2SB_X - 2\theta_X BSA = (\theta_{XYY} + 2\theta_{XXX})A. \tag{20}$$

Using (19) and (20), B can be in the following form

$$B = \left(\theta_{XX} + \frac{1}{2}\theta_{YY}\right)SA + \left(\psi_X + \frac{1}{4}\theta_X^2 + \frac{1}{4}\theta_Y^2\right)S + (\theta_X\theta_Y - \psi_Y)I, \tag{21}$$

where ψ and θ have the following relation

$$\psi_{XX} - \psi_{YY} = \frac{3}{4}\left(\theta_X^2 - \theta_Y^2\right)_X. \tag{22}$$

5. The KR and mNVN equations: The Pfaffian technique

In this section we will prove that Pfaffians satisfy the KR and mNVN equations.

The bilinear form of KR equations can be written in the following way

$$u = u_0 + i\ln(G/F), \qquad v = v_0 + \ln(GF), \tag{23}$$

where G and F are complex conjugates of one another. Introducing this change into (15) we get

$$v_{0xy} - u_{0x}u_{0y} + (FG)^{-1}\left[D_x D_y - i(u_{0x}D_y + u_{0y}D_x)\right]G\cdot F = 0. \tag{24}$$

We suppose that (u_0, v_0) is itself a solution of (14), (15) and so from (24) we find the bilinear equation

$$\left[D_x D_y - i(u_{0x}D_y + u_{0y}D_x)\right]G\cdot F = 0. \tag{25}$$

Now considering (14) in a similar way we get

$$u_{0xyt}+u_{0x}v_{0yt}+u_{0y}v_{0xt}+(FG)^{-1}\big[i(D_xD_yD_t+v_{0xt}D_y+v_{0yt}D_x)+u_{0x}D_yD_t+u_{0y}D_xD_t\big]G\cdot F$$
$$+(FG)^{-2}(-iD_tG\cdot F)\big[D_xD_y-i(u_{0x}D_y+u_{0y}D_x)\big]G\cdot F=0.$$

Since we suppose that (u_0, v_0) satisfies (14), and using (25), we get a second bilinear equation

$$\big[D_xD_yD_t+v_{0xt}D_y+v_{0yt}D_x-i(u_{0x}D_yD_t+u_{0y}D_xD_t)\big]G\cdot F=0. \tag{26}$$

The pair (25), (26) is the Hirota form of (14), (15). Particularly, if we take $u_0=0$, $v_{0xt}=\lambda$ and $v_{0yt}=\mu$, then the Hirota form (25), (26) simplifies to become

$$(D_xD_yD_t+\lambda D_y+\mu D_x)G\cdot F=0,$$
$$D_xD_yG\cdot F=0.$$

We introduce Pfaffians denoted by $(1,2,\dots,2n)$ which represent the functions G and F in the following form

$$G=(1,2,\dots,2n) \tag{27}$$

and

$$F=(1,2,\dots,2n)^* \tag{28}$$

whose elements are given by the skew-product

$$S[\theta_i,\theta_j]=\int_{(x_0,y_0)}^{(x,y)} W_x[\theta_i,\theta_j]\,dx-W_y[\theta_i,\theta_j]\,dy+\big(\theta_i^*(\theta_j)_t-(\theta_i)_t\theta_j^*\big)\,dt, \tag{29}$$

where $*$ denotes the complex conjugate, for $X=x$ or y or t, $W_X[a,b]=ab_X-a_Xb$ and

$$\theta_k=\phi_k^1+i\phi_k^2$$

in which, for $k=1,\dots,2n$, ϕ_k^1 and ϕ_k^2 satisfy Eqs. (10)–(13). The integral in (29) is written so that it is exact, thus

$$W_{xy}[\theta_i,\theta_j]=-W_{yx}[\theta_i,\theta_j],\qquad W_{xt}[\theta_i,\theta_j]=\big(\theta_i^*(\theta_j)_t-(\theta_i)_t\theta_j^*\big)_x,$$
$$W_{yt}[\theta_i,\theta_j]=-\big(\theta_i^*(\theta_j)_t-(\theta_i)_t\theta_j^*\big)_y.$$

Therefore, we may find the t dependent element in (29) by using these equalities with the linear equations given in (10)–(13). The (i,j)th element of Pfaffians in (27), (28) is $(i,j)=S[\theta_i,\theta_j]$ and $(I^i,j)=\phi_j^i$, $(I^i,I^j)=0$. Thus the (i,j)th element can be written in the following form

$$(i,j)=\int\big(W_x[\phi_i^1,\phi_j^1]-W_x[\phi_i^2,\phi_j^2]\big)\,dx-\big(W_y[\phi_i^1,\phi_j^1]-W_y[\phi_i^2,\phi_j^2]\big)\,dy+\big(W_t[\phi_i^1,\phi_j^1]+W_t[\phi_i^2,\phi_j^2]\big)\,dt$$
$$+i\big(\big(W_x[\phi_i^1,\phi_j^2]+W_x[\phi_i^2,\phi_j^1]\big)\,dx-\big(W_y[\phi_i^1,\phi_j^2]+W_y[\phi_i^2,\phi_j^1]\big)\,dy+\big(\phi_i^1\phi_j^2-\phi_i^2\phi_j^1\big)_t\,dt\big).$$

In order to prove Eqs. (14), (15) we exploit the identities of Pfaffians which correspond to the Jacobi identity of determinants, and show that the derivatives of the Pfaffians are represented by the sum of the Pfaffians. From (1) and (2), for $m=2$, we have the following Pfaffian identities

$$\begin{aligned}(a_1,a_2,a_3,a_4,b_1,b_2,\dots,b_{2m})(b_1,b_2,\dots,b_{2m})&=(a_1,a_2,b_1,b_2,\dots,b_{2m})(a_3,a_4,b_1,b_2,\dots,b_{2m})\\&\quad-(a_1,a_3,b_1,b_2,\dots,b_{2m})(a_2,a_4,b_1,b_2,\dots,b_{2m})\\&\quad+(a_1,a_4,b_1,b_2,\dots,b_{2m})(a_2,a_3,b_1,b_2,\dots,b_{2m})\end{aligned} \tag{30}$$

and

$$\begin{aligned}(a_1,a_2,a_3,b_1,b_2,\dots,b_{2m-1})(b_1,b_2,\dots,b_{2m})&=(a_1,b_1,b_2,\dots,b_{2m-1})(a_2,a_3,b_1,b_2,\dots,b_{2m})\\&\quad-(a_2,b_1,b_2,\dots,b_{2m-1})(a_1,a_3,b_1,b_2,\dots,b_{2m})\\&\quad+(a_3,b_1,b_2,\dots,b_{2m-1})(a_1,a_2,b_1,b_2,\dots,b_{2m}).\end{aligned} \tag{31}$$

With these properties we can also prove that the KR and mNVN equations reduce to the identity of Pfaffians. To write the derivatives of the Pfaffians one can introduce the symbol ∂_Z^j such that $(\partial_Z^j, i)=\phi_{iZ}^j$ where $Z=x,y,t,xy,xt,yt,xyt$, etc., and $(\partial_Z^i,\partial_Z^j)=0$. After substituting these results into the left-hand side of (25), which vanishes by virtue of the Pfaffian identity of the form (30). Similarly substituting these derivatives into the left-hand side of (26), vanishes by virtue of two Pfaffian identities of the form (31).

6. Conclusion

In this paper, we have elucidated the role of Pfaffians in determining new solutions to KR and mNVN equations. We have presented Pfaffian solutions to the bilinear KR and mNVN equations by using the Pfaffian technique [2]. In addition, we have obtained a new Lax pair for the mNVN equations. The Pfaffian technique can be applied to various integrable nonlinear evolution equations.

Acknowledgment

The author expresses his sincere thanks for the reviewer's suggestions and comments.

References

[1] E.R. Caianiello, Combinatorics and Renormalization in Quantum Field Theory, Benjamin, Reading, MA, 1973.
[2] R. Hirota, Soliton solutions to the BKP equations. I. The Pfaffian technique, J. Phys. Soc. Japan 58 (1989) 2285–2296.
[3] B. Konopelchenko, C. Rogers, On $(2+1)$-dimensional nonlinear-systems of Loewner-type, Phys. Lett. A 158 (1991) 391–397.
[4] J.J.C. Nimmo, A class of solutions of the Konopelchenko–Rogers equations, Phys. Lett. A 168 (1992) 113–119.
[5] J.J.C. Nimmo, Darboux transformations in $(2+1)$-dimensions, in: P.A. Clarkson (Ed.), Applications of Analytic and Geometric Methods to Nonlinear Differential Equations, Kluwer, Dordrecht, 1993, pp. 183–192.
[6] J.J.C. Nimmo, Darboux transformations from reductions of the KP hierarchy, in: Vladimir G. Makhankov, Alan R. Bishop, Darryl D. Holm (Eds.), Nonlinear Evolution Equations & Dynamical Systems Needs'94, World Scientific, Singapore, 1995.